Maths Journal - Polynomials

***Royally Certified Mathematics Journal (Please refer to the sources at the end of this journal to prove the accuracy and certification of this journal)

Syllabus Dot Points - Polynomials (ME-F2):

- uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses ME11-1
- manipulates algebraic expressions and graphical functions to solve problems ME11-2
- uses appropriate technology to investigate, organise and interpret information to solve problems in a range of contexts ME11-6
- communicates making comprehensive use of mathematical language, notation, diagrams and graphs ME11-7

Content:

F2.1: Remainder and factor theorems

- define a general polynomial in one variable, x, of degree n with real coefficients to be the expression: $anxn + an-1xn-1 + \cdots + a2x2 + a1x + a0$, where $an \neq 0$
    - understand and use terminology relating to polynomials including degree, leading term, leading coefficient and constant term
- use division of polynomials to express $P(x)$ in the form $P(x) = A(x).\ Q(x) + R(x)$ where deg $R(x) <$ deg $A(x)$ and $A(x)$ is a linear or quadratic divisor, $Q(x)$ the quotient and $R(x)$ the remainder
    - review the process of division with remainders for integers
    - describe the process of division using the terms: dividend, divisor, quotient, remainder
- prove and apply the factor theorem and the remainder theorem for polynomials and hence solve simple polynomial equations (ACMSM089, ACMSM091)

F2.2: Sums and products of roots of polynomials

- solve problems using the relationships between the roots and coefficients of quadratic, cubic and quartic equations AAM
    - consider quadratic, cubic and quartic equations, and derive formulae as appropriate for the sums and products of roots in terms of the coefficients
- determine the multiplicity of a root of a polynomial equation
    - prove that if a polynomial equation of the form $P(x) = 0$ has a root of multiplicity $r > 1$, then $P'(x) = 0$ has a root of multiplicity $r - 1$

- graph a variety of polynomials and investigate the link between the root of a polynomial equation and the zero on the graph of the related polynomial function
    - examine the sign change of the function and shape of the graph either side of roots of varying multiplicity

Lesson 1: Division of Polynomials

Syllabus Dot Point to cover:

- define a general polynomial in one variable, x, of degree n with real coefficients to be the expression: $anxn + an-1xn-1 + \cdots + a2x2 + a1x + a0$, where $an \neq 0$
    - understand and use terminology relating to polynomials including degree, leading term, leading coefficient and constant term
- use division of polynomials to express $P(x)$ in the form $P(x) = A(x).\ Q(x) + R(x)$ where deg $R(x) <$ deg $A(x)$ and $A(x)$ is a linear or quadratic divisor, $Q(x)$ the quotient and $R(x)$ the remainder
    - review the process of division with remainders for integers
    - describe the process of division using the terms: dividend, divisor, quotient, remainder

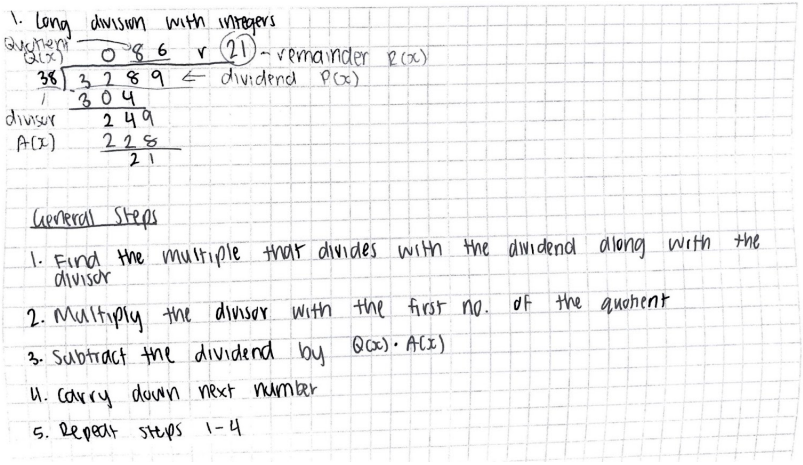
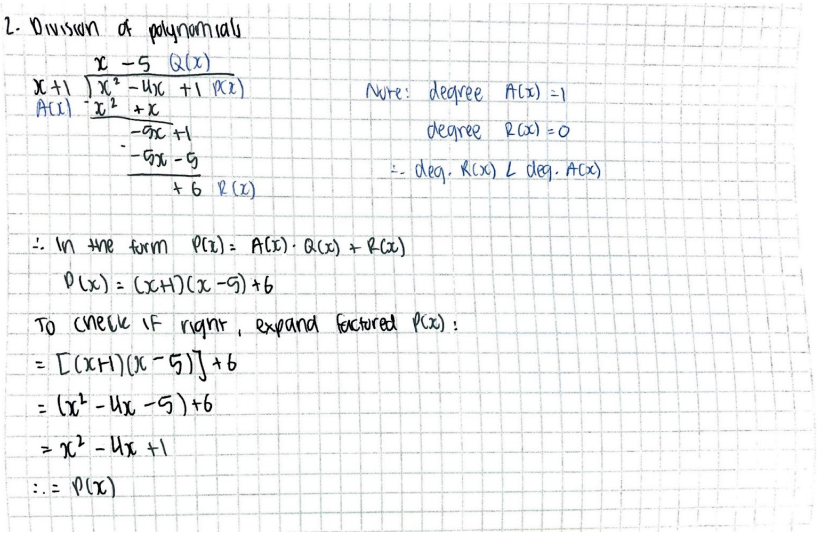
Learning Goals:

- ☑ ~~Define what a general polynomial is~~
- ☑ ~~To express P(x) in the form of P(x)= A(x).Q(x) + R(x)~~
- ☑ ~~Understand, and link that Deg. R(x) < deg. A(x)~~
- ☑ ~~Revise on the process of long division with integers~~
- ☑ ~~Name each part of the long division and express it in form~~

Definition of a General Polynomial

The general form of a polynomial can be expressed as $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where $a_n \neq 0$ where it can be seen that the first term is the leading term, where *a* is the leading coefficient, *n* is the degree, and *a0* is the constant term.

Questions

1. Long division with integers	2. Division of Polynomials
	
3. Understanding Remainders	4. Harder division of Polynomials

2 Use long division to perform each division. Express each result in the form $P(x) = D(x)\,Q(x) + R(x)$.

a $(x^2 - 4x + 1) \div (x + 1)$

5 **a** If the divisor of a polynomial has degree 3, what are the possible degrees of the remainder?

b On division by $D(x)$, a polynomial has remainder $R(x)$ of degree 2. What are the possible degrees of $D(x)$?

3. Understanding Remainders

a) Since deg. R(x) < deg. A(x)

degree R(x) < 3

∴ degree of R(x) can be = 2, 1, 0

b) Since deg. R(x) < deg. A(x)

degree A(x) > 2

∴ degree of A(x) can be = 3, 4, 5, 6 ... ∞

11 If $x^4 - 2x^3 - 20x^2 + mx + n$ is exactly divisible by $x^2 - 5x + 2$, find m and n.

First attempt

(b) If $x^4 - 2x^3 - 20x^2 + mx + n$ is exactly divisible by $x^2 - 5x + 2$, find m & n.

$x^2 - 5x + 2 = 0$

$x = \frac{-(-5) \pm \sqrt{5^2 - 4(1)(2)}}{2(1)}$

$= \frac{5 \pm \sqrt{25 - 8}}{2}$

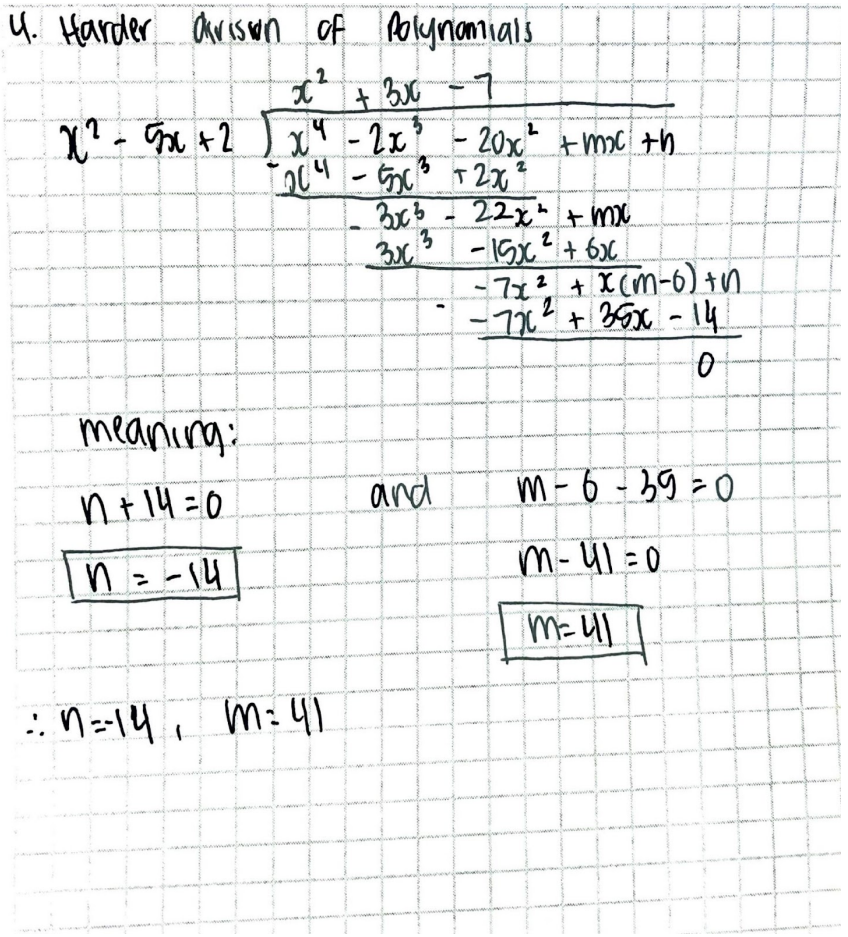
$= \frac{5 \pm \sqrt{17}}{2}$

$P(x) = x^4 - 2x^3 - 20x^2 + mx + n$

$P\left(\frac{5+\sqrt{17}}{2}\right) = x^4 - 2x^3 - 20x^2 + mx + n$

$= \left(\frac{5+\sqrt{17}}{2}\right)^4 - 2\left(\frac{5+\sqrt{17}}{2}\right)^3 - 20\left(\frac{5+\sqrt{17}}{2}\right)^2 + m\left(\frac{5+\sqrt{17}}{2}\right) + n$

Second attempt



Further Study

From watching the following videos by Eddie Woo, some key information that I've found include:

1. The general steps that include in division include:
    a. Dividing
    b. Multiplying
    c. Subtracting
    d. Carry down
2. When dividing a polynomial, you only need to divide by the leading term
3. In the multiplying step, although you have only divided with the leading term, you need to multiply the quotient by the whole divisor and not just the leading term
4. The reasons as to why polynomial division is important is because:
    a. It gives insight to what polynomials are doing, and why they do what they do
    b. It helps with integration, and is one of the key tools that help with integration
    c. It illustrates how we as mathematicians work with non-numerical values and still be able to identify and observe trends

Video links:

Intro to Polynomial Division (1 of 3: Comparing to numerical division)

Intro to Polynomial Division (2 of 3: Basic example)

Intro to Polynomial Division (3 of 3: Why is it important?)

Conclusion

From this lesson, the main thing that I've learnt is the reason as to why we divide polynomials, and what does it help with. This is because, as students, we learn **how** we implement that concept, but don't really understand **why** we learn that concept, and what it helps with. It helped me to understand that polynomial division is very useful in aspects such as finding unknown values of pronumerals. As such, through further studying of watching Eddie Woo videos, I am now able to recognise how polynomial division assists with key things in calculus, As a whole, this lesson has helped me realise the purpose of dividing polynomials, and how it helps with how we understand algebra as a whole.

Lesson goals: Achieved
Syllabus Outcomes: Achieved

Lesson 2: Remainder Theorem

Syllabus Dot Points:

- Prove and apply the factor theorem and the remainder theorem for polynomials and hence solve simple polynomial equations (ACMSM089, ACMSM091)

Learning Goals:

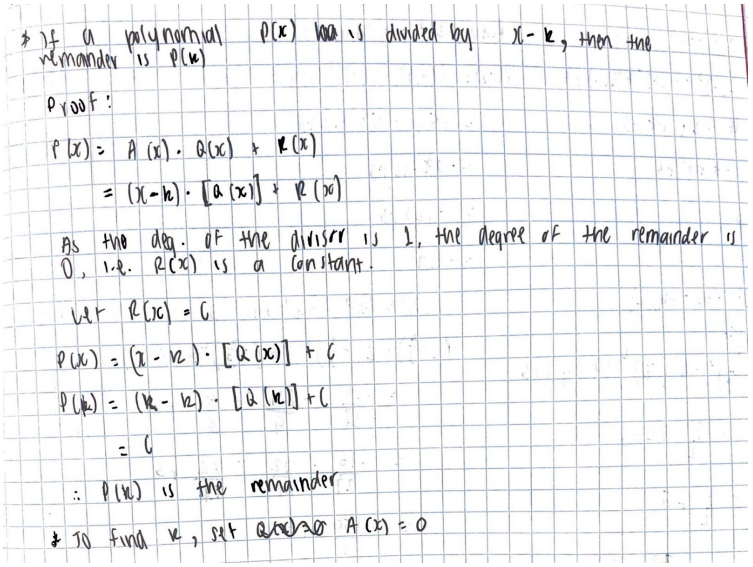
- ☑ ~~To derive the remainder theorem~~
- ☑ ~~To understand how to use the remainder theorem to solve simple polynomial equations~~
- ☑ ~~To understand the correlation between dividing polynomials and doing the remainder theorem~~
- ☑ ~~State what the remainder theorem is~~

Definition of the Remainder Theorem

The remainder theorem states that when f(x) is divided by a (x-a), then the remainder will be f(a).

Sources: Remainder Theorem definition

Derivation of the Remainder Theorem



Questions

1. Understanding the correlation between polynomial division and remainder theorem	2. Remainder theorem in simple polynomial equations

1 Without division, find the remainder when $P(x) = x^3 - x^2 + 2x + 1$ is divided by:

a $x - 1$

1. Understanding the correlation ... remainder theorem

$x - 1 \overline{)\, x^3 - x^2 + 2x + 1}$

x^2

$x^2 + 2$

$x - 1 \overline{)\, x^3 - x^2 + 2x + 1}$

$-x^3 - x^2$

$0 + 2x + 1$

$+2x - 2$

$0 + 3$

$\therefore P(x) = (x-1)(x^2+2) + 3$

Now using remainder theorem:

$P(x) = x^3 - x^2 + 2x + 1$

$P(1) = (1)^3 - (1)^2 + 2(1) + 1$

$= 3$

$\therefore$ shows the correlation between Polynomial division and the remainder theorem.

3. Harder Polynomial Questions

2. Remainder theorem in simple polynomial eqns

$P(x) = 2x^3 - x^2 + px - 1$ divisor: $x - 3$ $R(x) = 2$

$x - 3 \overline{)\, 2x^3 - x^2 + px - 1}$

$P(x) = 2x^3 - x^2 + px - 1$ $A(x) = x - 3$ $R(x) = 2$

$P(x) = 2x^3 - x^2 + px - 1$

$P(3) = 2(3)^3 - (3)^2 + p(3) - 1$

$2 = 54 - 9 + 3p - 1$

$2 = 44 + 3p$

$-42 = 3p$

$\boxed{p = -14}$

4. Harder Polynomial Questions

3. Harder Polynomial Questions

14.

a) This is because deg. $R(x) < \text{deg. } A(x)$,

Since deg. $A(x) = 2$,

deg. $R(x) < 2$, meaning has to be a linear equation,

$\therefore$ is in the form $ax + b$

b) $P(x) = A(x) \cdot Q(x) + R(x)$

$P(1) = (x-1)(x+2) \cdot Q(x) + ax + b$

$2 = (x^2 + x - 2) \cdot Q(x) + ax + b$

$2 = [(1)^2 + (1) - 2] \cdot Q(x) + ax + b$

$2 = ax + b$ (1)

$2 = a(1) + b$

$2 = a + b$ ①

$P(-2) = (x-1)(x+2) \cdot Q(x) + ax + b$

$5 = -2a + b$ ②

$$\begin{array}{rl} a+b &= 2 \\ -2a+b &= 5 \end{array} \quad \text{— ② - ①}$$

$3a = 7$

$\boxed{a = \frac{7}{3}}$

$3a = -3$

$a = -1$

Sub $a = -1$ into ①

$2 = (-1) + b$

$\boxed{b = 3}$

(continued...)

$\therefore a = -1, \ b = 3$

4. Harder Polynomial Question

16 c)

$A(x) = x^2 - 5$

$R(x) = x + 4$

$P(x) = A(x) \cdot Q(x) + R(x)$

$P(x) = (x^2 - 5) \cdot Q(x) + x + 4$

$P(-x) = (x^2 - 5) \cdot Qx - x + 4$

$P(x) + P(-x) = [(x^2 - 5) \cdot Q(x)] + x + 4 + [(x^2 - 5) \cdot Q(x)] - x + 4$

$= 2[(x^2 - 5) \cdot Q(x)] + 8$

$\boxed{R(x) = 8}$

↑ remainder

Further Study

1. If the divisor has a degree that is larger than 1, then the remainder will be a function of x
2. The remainder theorem is a shortcut method of finding the remainder in that it helps to minimise the working out needed to find it
3. It helps to factorise polynomials, as finding out if there is a remainder will help to find what are the factors of the polynomial

Video links:
Remainder & Factor Theorem (1 of 3: The division transformation)
Remainder & Factor Theorem (2 of 3: Fully factorising a polynomial)
Remainder & Factor Theorem (3 of 3: Determining a polynomial from remainders)

Conclusion

This lesson was more of a revision lesson, in that I was able to deepen my understanding of using the remainder theorem in harder polynomial questions, which hence therefore allowed me to find values of unknown pronumerals. It can be seen that I made quite a few silly mistakes, in that I was able to find out quickly, but I have to therefore be more careful about doing the questions.

Lesson goals: Achieved
Syllabus Outcomes: Achieved

Lesson 3: Factor Theorem

Syllabus Dot Points:

- prove and apply the factor theorem and the remainder theorem for polynomials and hence solve simple polynomial equations (ACMSM089, ACMSM091)

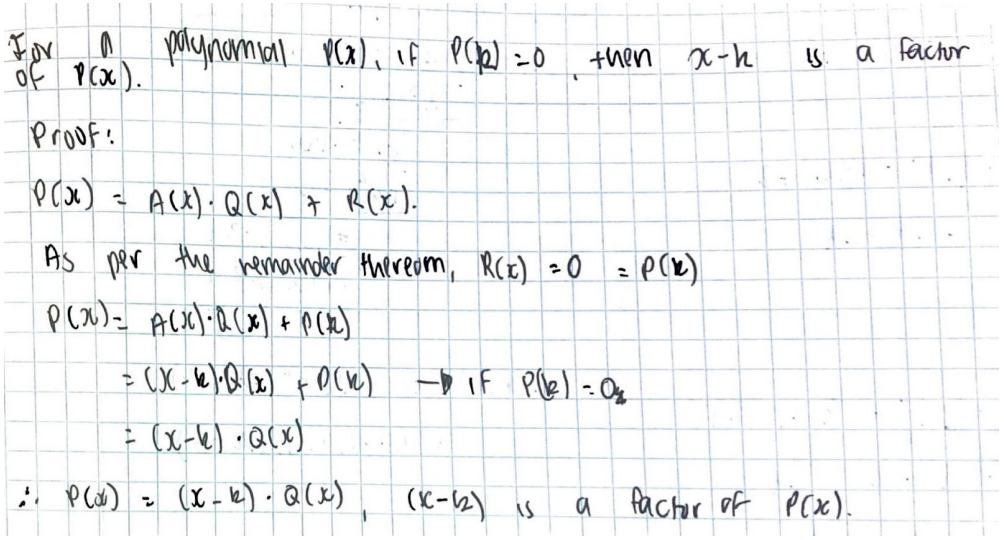
Learning Goals:

- ☑ ~~To derive the factor theorem~~
- ☑ ~~To understand the relationship between the remainder theorem and the factor theorem~~
- ☑ ~~State what the factor theorem is~~

Definition of Factor theorem, and the difference between remainder theorem and factor theorem

The definition of the factor theorem is that if P(k) = 0, then (x-k) is a factor of P(x)

Derivation of Factor theorem



Questions

1. Understanding the correlation between factor theorem and remainder theorem	2. Understanding how factor theorem helps with Polynomial sketching
2 Without division, find which of the following are factors of $F(x) = x^3 + 4x^2 + x - 6$.	**4** Factor each polynomial and sketch its graph, indicating all intercepts with the axes. Remember that any integer zeroes are divisors of the constant term.
a $x - 1$	**a** $P(x) = x^3 + 2x^2 - 5x - 6$

1.

Is $(x-1)$ a factor of $P(x) = x^3 - x^2 + 2x + 1$

$P(x) = x^3 - x^2 + 2x + 1$

$P(1) = (1)^3 - (1)^2 + 2(1) + 1$

$= 3$

∴ Since remainder $\neq 0$, $(x-1)$ is not a factor of $P(x)$

2.

(a)

(i) $P(x) = x^3 + 2x^2 - 5x - 6$

If $(x+2)$ is a factor of $P(x)$, then $P(-2) = 0$

$P(-2) = (-2)^3 + 2(-2)^2 - 5(-2) - 6$

$= -8 + 8 + 10 - 6$

$= 4$

If $(x+1)$ is a factor of $P(x)$, then $P(-1) = 0$

$P(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$

$= -1 + 2 + 5 - 6$

$= 0$

∴ $(x+1)$ is a factor of $P(x)$.

be careful of multiplying & adding.

If $(x-1)$ is a factor of $P(x)$, then $P(1) = 0$

$P(1) = (-1)^3 + 2(1)^2 - 5(1) - 6$

$= -1 + 2 - 5 - 6$

$= -10 \neq 0$

If $(x-2)$ is a factor of $P(x)$, then $P(2) = 0$

$P(2) = (2)^3 + 2(2) - 5(2) - 6$

$= 8 + 4 - 10 - 6$

$= -4$

unnecessary.

If $(x-3)$ is a factor of $P(x)$, then $P(3) = 0$

$P(3) = (3)^3 + 2(3) - 5(3) - 6$

$= 27 + 6 - 15 - 6$

If $(x+1)$ is a factor then:

$$x+1 \overline{)\, x^3 + 2x^2 - 5x - 6}$$

$x^2 + x - 6$

$-(x^3 + x^2)$

$x^2 - 5x$

$-(x^2 + x)$

$-6x - 6$

$-(-6x - 6)$

0

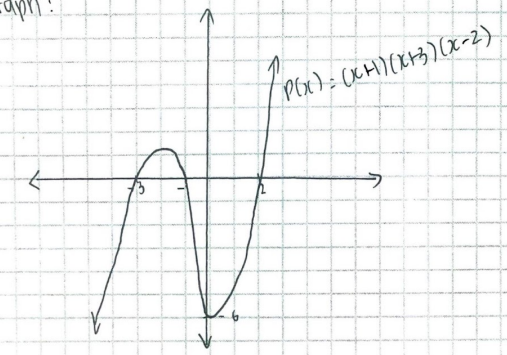
$\therefore P(x) = (x+1)(x^2 + x - 6)$

$= (x+1)(x+3)(x-2)$

roots: $-1, -3, 2$

y-int: -6

Graph:



3. Questions

5 Solve these equations by first factoring the LHS.

c $6x^3 - 5x^2 - 12x - 4 = 0$

4. Questions

12 a $P(x)$ is an odd polynomial of degree 3. It has $x + 4$ as a factor, and when it is divided by $x - 3$ the remainder is 21. Find $P(x)$.

(12)

a) ~~$P(x) = A(x) \cdot Q(x) + R(x)$~~

$= (x-3) \cdot Q(x) + 21$

$P(x) = A(x) \cdot Q(x) + R(x)$

$= (x+4) \cdot Q(x)$

~~$(x-3)Q(x) + 21 = (x+4) \cdot Q(x)$~~

~~$P(x) = A(x) \cdot Q(x) + R(x)$~~

$= (x+4) \cdot Q(x)$

$P(3) = (3+4) \cdot Q(3)$

$21 = 7 \cdot Q(3)$

~~$4 = Q(3)$~~

consider general form of cubic:

$ax^3 + bx^2 + cx + d$

since $P(x)$ is an odd function, $P(-x) = -P(x)$

$\therefore a(-x)^3 + b(-x)^2 + c(-x) + d = -(ax^3 + bx^2 + cx + d)$

$-ax^3 + bx^2 - cx + d = -ax^3 - bx^2 - cx - d$

$2bx^2 + 2d = 0$

$bx^2 + d = 0$

or

consider another form of cubic:

$P(x) = ax^3 + cx$

$P(-4) = a(-4)^3 + c(-4)$

$0 = -64a - 4c$

$0 = -4(16a + c)$

$c = -16a$

$P(x) = ax^3 + (-16a)x$

$= ax^3 - 16ax$

$P(3) = a(3)^3 - 16a(3)$

$21 = 27a - 48a$

$21 = -21a$

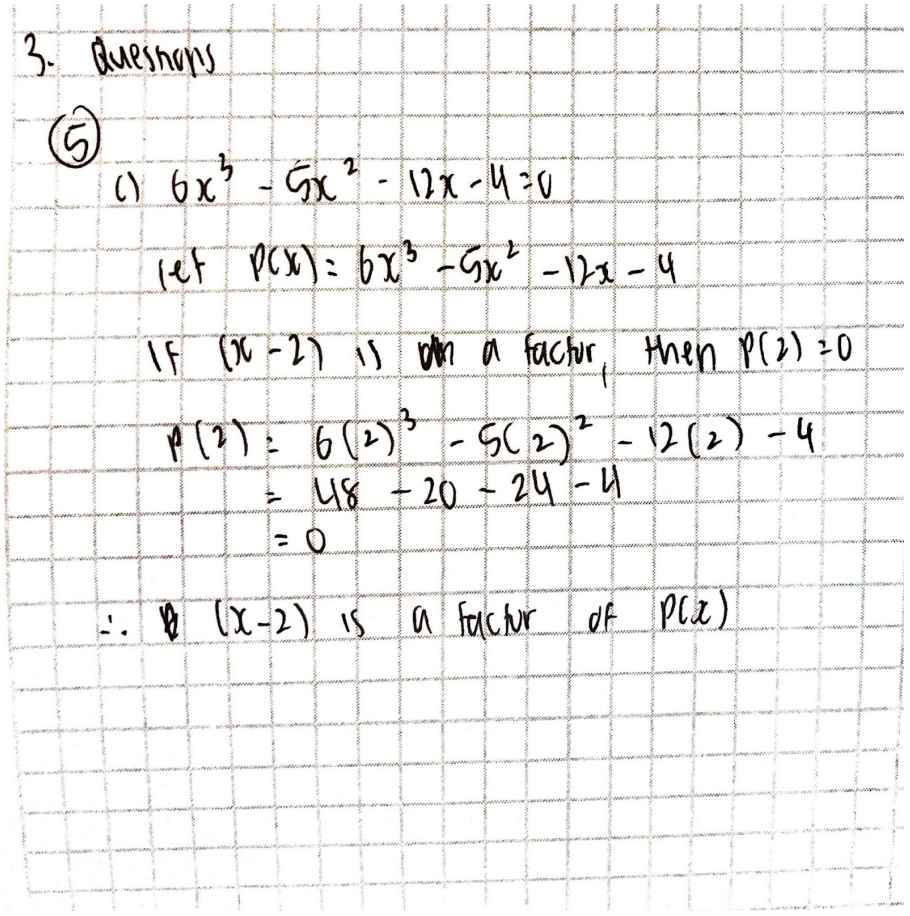
$\boxed{a = -1}$

$P(x) = -1ax^3 + cx$

$= (-1)x^3 + (-16(-1))x$

$= -x^3 + 16x$

$\therefore P(x) = -x^3 + 16x$



3. Questions

(5)

(i) $6x^3 - 5x^2 - 12x - 4 = 0$

let $P(x) = 6x^3 - 5x^2 - 12x - 4$

If $(x-2)$ is a factor, then $P(2) = 0$

$P(2) = 6(2)^3 - 5(2)^2 - 12(2) - 4$

$= 48 - 20 - 24 - 4$

$= 0$

$\therefore$ $(x-2)$ is a factor of $P(x)$

Help source: https://qr.ae/pvoqDD

$$
\begin{array}{r|l}
 & 6x^2 + 7x + 2 \\
\hline
x-2 & 6x^3 - 5x^2 - 17x - 4 \\
 & -6x^3 - 12x^2 \\
\hline
 & \quad 7x^2 - 17x \\
 & \quad -7x^2 - 14x \\
\hline
 & \qquad -3x - 4 \\
 & \qquad -3x - 4 \\
\hline
 & \qquad\quad 0
\end{array}
$$

$\therefore P(x) = (x-2)(6x^2 + 7x + 2)$

$= (x-2)[6x^2 + 3x + 4x + 2]$

$= (x-2)[3x(2x+1) + 2(2x+1)]$

$= (x-2)(3x+2)(2x+1)$

Since $P(x) = 0$

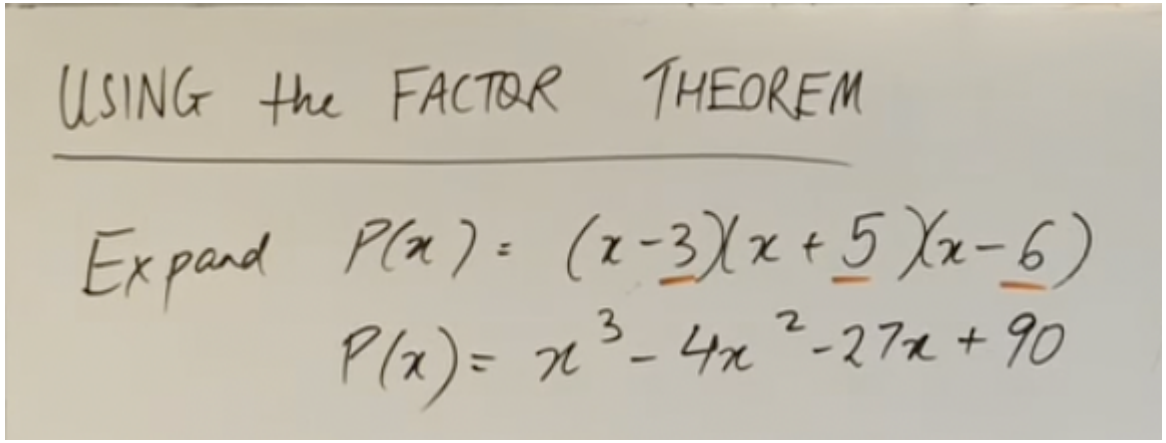
$(x-2)(3x+2)(2x+1) = 0$

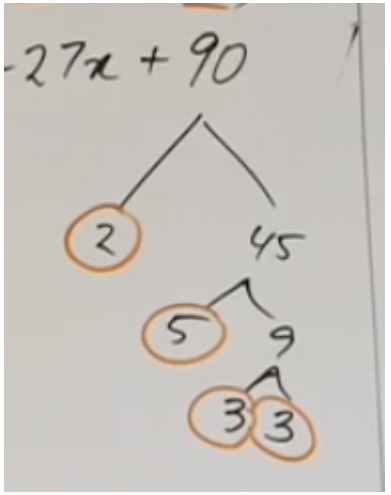
$\therefore \boxed{x = 2, -\frac{2}{3}, -\frac{1}{2}}$

Further Study

Using the Factor Theorem

1. When factoring polynomials, it is very hard to factorise a polynomial in the form that is P(x) = ax^3+bx^2+cx+d. It allows you to not “randomly guess” which numbers would be a factor of the polynomial. Instead, you would use numbers that are factors of the constant term, to see if it does divide easily





It can be seen that the factors of this polynomial when multiplied give the constant term, so that means that there is a higher chance that a number that is a factor of the constant term will become a factor of the polynomial. Using the tree diagram will allow it to find the most reoccurring factor that would give a higher chance for something to be factor

Conclusion

This lesson overall was quite a revision lesson, although it had allowed me to explore many ways of using the factor theorem, and explore how it can be useful in saving time during exam times.

Syllabus Outcomes: Achieved
Learning Goals: Achieved

Lesson 4: Sum and Product of Roots

Syllabus Points:

- solve problems using the relationships between the roots and coefficients of quadratic, cubic and quartic equations AAM
    - consider quadratic, cubic and quartic equations, and derive formulae as appropriate for the sums and products of roots in terms of the coefficients

Learning Goals:

- ☑ ~~Derive the sum and product of roots of quadratic, cubic and quartic equations~~
- ☑ ~~To understand the basic concept of the sum and product of roots~~
- ☑ ~~Find the purpose of the sum and product of roots and what does it actually mean and tell us~~

Derivation of Sum and Product of Roots

Quadratics

Quadratics:

Let quadratic be $= ax^2 + bx + c = 0$, where α & β are roots

$\therefore (x - \alpha)(x - \beta) = 0$

$x^2 - x\beta - x\alpha + \alpha\beta = 0$

$x^2 - x(\beta + \alpha) + \alpha\beta = 0$

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

↓ closely mimics form

$ax^2 + bx + c = 0$

$\therefore$ can express $ax^2 + bx + c = 0$ in monic form as:

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\therefore -(\alpha+\beta) = \frac{b}{a}$

$\boxed{\alpha+\beta = -\frac{b}{a}}$

$\boxed{\alpha\beta = \frac{c}{a}}$

Sum of roots:

$\alpha+\beta = -\frac{b}{a}$

product of roots:

$\alpha\beta = \frac{c}{a}$

Useful identity

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2(\alpha\beta)$

RHS: $(\alpha+\beta)^2 - 2(\alpha\beta)$

$= [\alpha^2 + 2\alpha\beta + \beta^2] - 2\alpha\beta$

$= \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta$

$= \alpha^2 + \beta^2$

= LHS

= as required.

Cubics

Cubics:

Let cubic eqn $= ax^3 + bx^2 + cx + d = 0$ that have roots α, β, γ

$\therefore (x-\alpha)(x-\beta)(x-\gamma) = 0$

$(x^2 - \alpha x - \beta x + \alpha\beta)(x-\gamma) = 0$

$x^3 - \alpha x^2 - \beta x^2 + \alpha\beta x - \gamma x^2 + \alpha\gamma x + \beta\gamma x - \alpha\beta\gamma = 0$

~~$x^3 - \alpha\beta\gamma(x^2)$~~

$x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma = 0$

We can express any cubic in monic form:

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$

Sum of roots:

$-(\alpha+\beta+\gamma) = \frac{b}{a}$

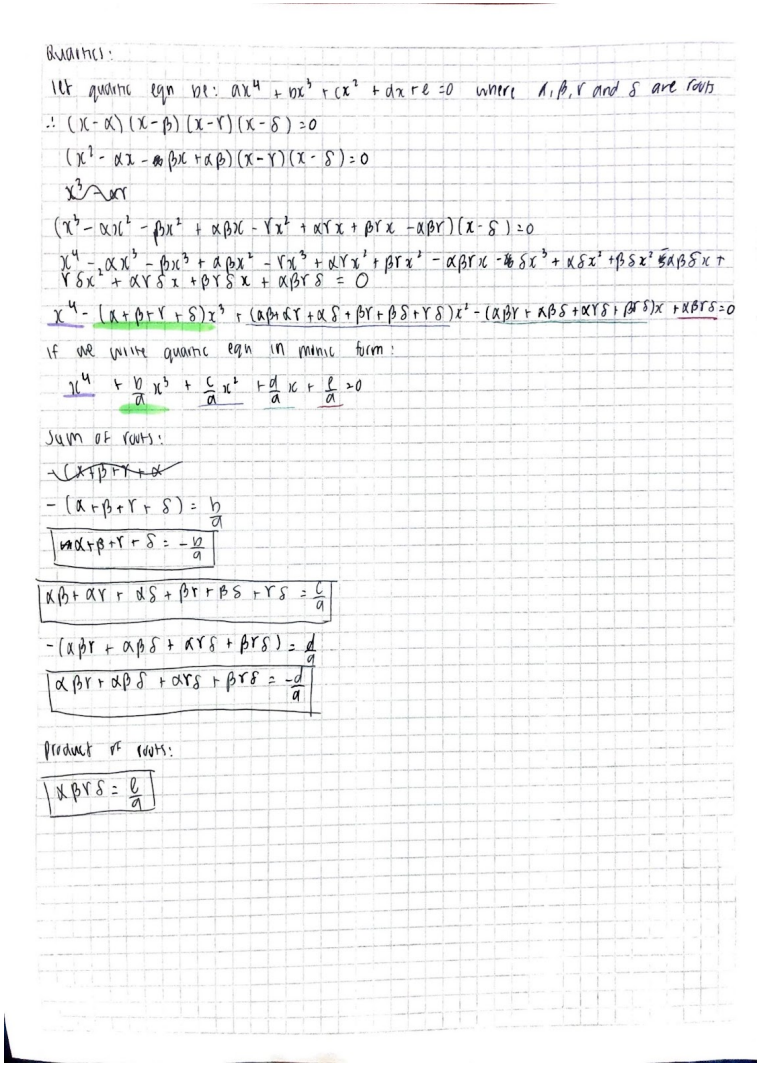
$\boxed{\alpha+\beta+\gamma = -\frac{b}{a}}$

$\boxed{\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}}$

$-\alpha\beta\gamma = \frac{d}{a}$ ← product of roots

$\boxed{\alpha\beta\gamma = -\frac{d}{a}}$

Quartics

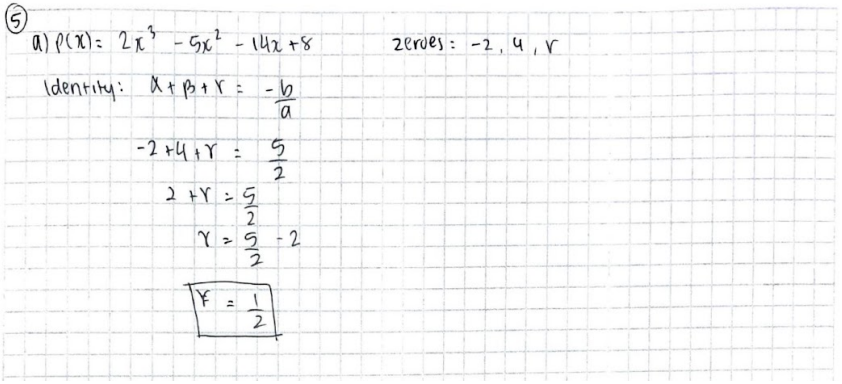

Quartics:
If quartic eqn is: $ax^4 + bx^3 + cx^2 + dx + e = 0$ where α, β, γ and δ are roots
$\therefore (x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0$
$(x^2 - \alpha x - \beta x + \alpha\beta)(x-\gamma)(x-\delta) = 0$
x^3
$(x^3 - \alpha x^2 - \beta x^2 + \alpha\beta x - \gamma x^2 + \alpha\gamma x + \beta\gamma x - \alpha\beta\gamma)(x-\delta) = 0$
$x^4 - \alpha x^3 - \beta x^3 + \alpha\beta x^2 - \gamma x^3 + \alpha\gamma x^2 + \beta\gamma x^2 - \alpha\beta\gamma x - \delta x^3 + \alpha\delta x^2 + \beta\delta x^2 - \alpha\beta\delta x + \gamma\delta x^2 + \alpha\gamma\delta x + \beta\gamma\delta x + \alpha\beta\gamma\delta = 0$
$x^4 - [\alpha+\beta+\gamma+\delta]x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + \alpha\beta\gamma\delta = 0$
If we write quartic eqn in monic form:
$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$
Sum of roots:
$-(\alpha+\beta+\gamma+\delta) = \frac{b}{a}$
$\alpha+\beta+\gamma+\delta = -\frac{b}{a}$
$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
$-(\gamma\delta + \beta\delta + \beta\gamma + \alpha\delta + \alpha\gamma + \alpha\beta) = \frac{d}{a}$
$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
Product of roots:
$\alpha\beta\gamma\delta = \frac{e}{a}$

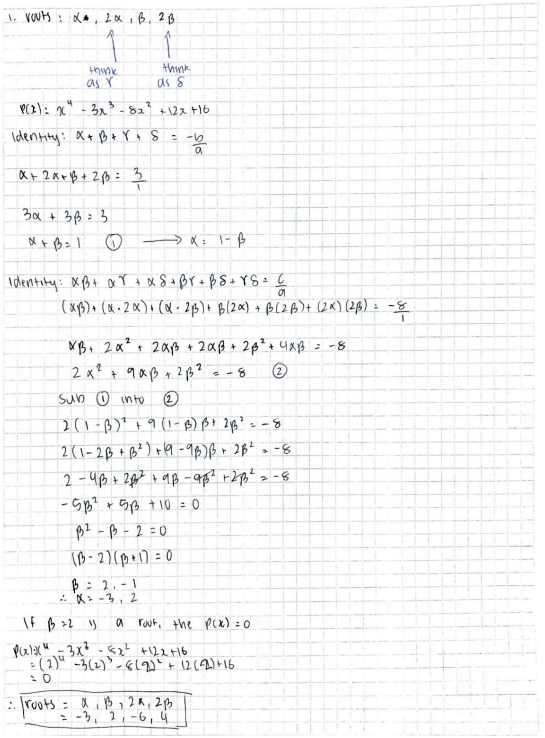
Questions



12 Find the zeroes of the polynomial $x^4 - 3x^3 - 8x^2 + 12x + 16$ if they are α, 2α, β and 2β.

5 a The polynomial $P(x) = 2x^3 - 5x^2 - 14x + 8$ has zeroes at -2 and 4. Use the sum of the zeroes to find the other zero.





b Find the values of a and b for which $x^2 - x - 20$ is a factor of $x^4 + ax^3 - 23x^2 + bx + 60$, and then find all the zeroes.

9 a If one of the roots of the equation $x^2 + bx + c = 0$ is twice the other root, show that $2b^2 = 9c$. (Hint: Let the roots be α and 2α.)

(8)

b) Let $P(x) = x^4 + ax^3 - 23x^2 + bx + 60$

Since $(x^2 - x - 20)$ is a factor:

$(x-5)(x+4) = (x^2 - x - 20)$

∴ 2 of the zeroes are $5, -4$

α β

$P(5) = (5)^4 + a(5)^3 - 23(5)^2 + b(5) + 60$

$0 = 625 + 125a - 575 + 5b + 60$

$0 = 125a + 5b + 110$

$0 = 25a + b + 22$

~~$25a + b = -110$~~

$25a + b = -22$ ①

$P(-4) = (-4)^4 + a(-4)^3 - 23(-4)^2 + b(-4) + 60$

~~$0 = 256 - 64a + 92 - 4b + 60$~~ ~~$60 = -64a - 4b + 408$~~

$0 = -64a - 4b - 104$

$0 = -16a - b - 26$

$0 = 16a + b + 26$

$16a + b = -26$ ②

① - ②

$25a + b = -22$

$16a + b = -26$

$9a = 4$

$a = \frac{4}{9}$

$0 = -16a - b + 102$

$102 = 16a + b$ ②

① - ②

$25a + b = -22$

$-16a + b = 102$

$9a = -124$

$a = \frac{-124}{9}$ ← looks wrong.

(9)

a) $x^2 + bx + c = 0$ roots: α, 2α

Identity: $\alpha + \beta = \frac{-b}{a}$

$3\alpha = \frac{-b}{1}$

$3\alpha = -b$ → $\alpha = \frac{-b}{3}$ ①

$\alpha\beta = \frac{c}{a}$

$\alpha \cdot 2\alpha = \frac{c}{a}$

$2\alpha^2 = \frac{c}{a}$

$2\alpha^2 = c$ ②

Sub ① into ②

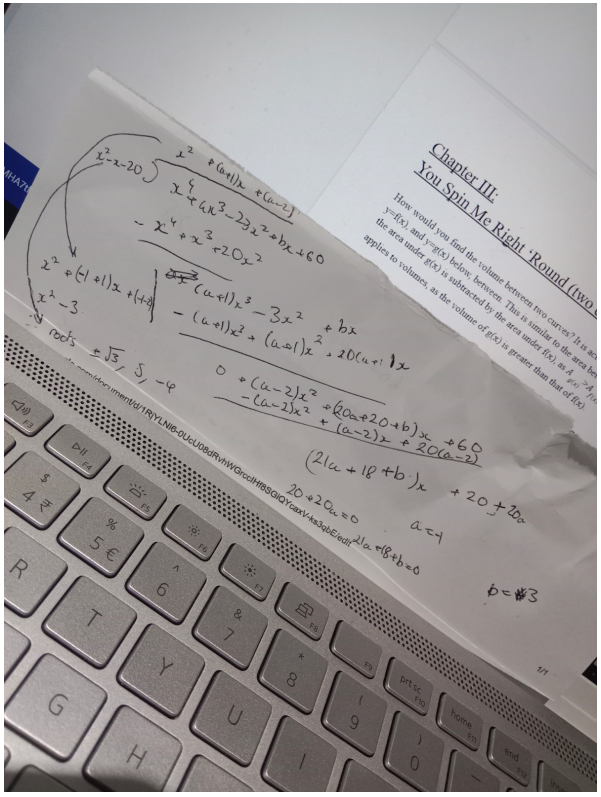
$2\alpha^2 = c$

$2\left(-\frac{b}{3}\right)^2 = c$

$2 \cdot \frac{b^2}{9} = c$

$2b^2 = 9c$

∴ as required

Conclusion

Through practising a number of these questions, I was able to attain a deeper understanding of the roots of polynomials, and didn't need assistance from Eddie Woo to explain this. I also feel more confident in these types of questions that allow me to gain some practice. Deriving each of the identities was very helpful as now I understand why these identities arise and now just how to use them in questions.

Syllabus Outcomes: Achieved
Learning Goals: Achieved

Lesson 5: Sums and Products of Cubics and Quartics

Syllabus Points:

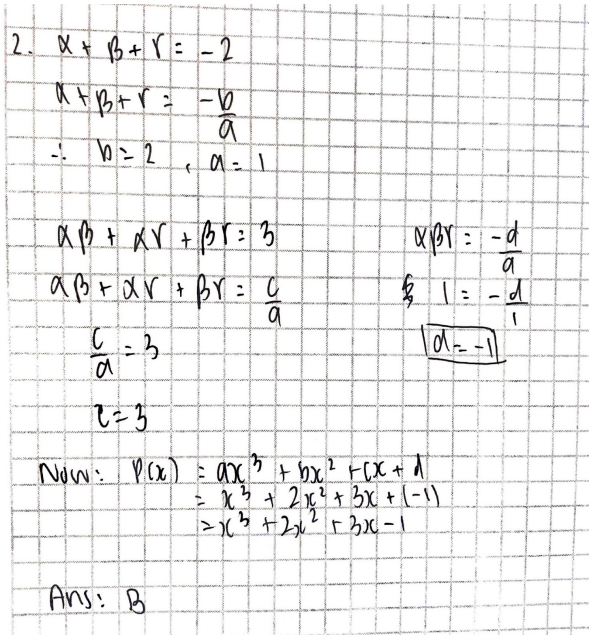
- solve problems using the relationships between the roots and coefficients of quadratic, cubic and quartic equations AAM
    - consider quadratic, cubic and quartic equations, and derive formulae as appropriate for the sums and products of roots in terms of the coefficients

Learning Goals:

- ☑ ~~To practice exam-style and past-hsc questions on cubic and quartic roots~~

Questions

2. Functions, EXT1 F2 2012 HSC 3 MC A polynomial equation has roots α, β, γ where $\alpha+\beta+\gamma=-2,\quad \alpha\beta+\alpha\gamma+\beta\gamma=3,\quad \alpha\beta\gamma=1.$ Which polynomial equation has the roots α, β, and γ? (A) $x^3+2x^2+3x+1=0$ (B) $x^3+2x^2+3x-1=0$ (C) $x^3-2x^2+3x+1=0$ (D) $x^3-2x^2+3x-1=0$	3. Functions, EXT1' F2 2012 HSC 5 MC The equation $2x^3-3x^2-5x-1=0$ has roots α, β and γ. What is the value of $\frac{1}{\alpha^3\beta^3\gamma^3}$? (A) $\frac{1}{8}$ (B) $-\frac{1}{8}$ (C) 8 (D) -8	13. Functions, EXT1 F2 2021 HSC 11h The roots of $x^4-3x+6=0$ are α, β, γ and δ. What is the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$? *(2 marks)*



2. $\alpha+\beta+\gamma=-2$

$\alpha+\beta+\gamma=-\frac{b}{a}$

$\therefore b=2$, $a=1$

$\alpha\beta+\alpha\gamma+\beta\gamma=3$

$\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$

$\frac{c}{a}=3$

$c=3$

$\alpha\beta\gamma=-\frac{d}{a}$

& $1=-\frac{d}{1}$

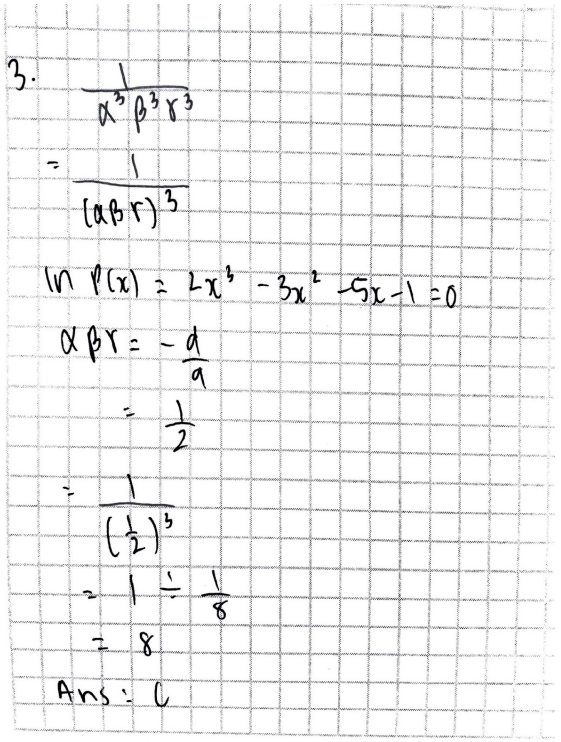
$d=-1$

Now: $P(x)=ax^3+bx^2+cx+d$

$=x^3+2x^2+3x+(-1)$

$=x^3+2x^2+3x-1$

Ans: B



3. $\frac{1}{\alpha^3\beta^3\gamma^3}$

$=\frac{1}{(\alpha\beta\gamma)^3}$

In $P(x)=2x^3-3x^2-5x-1=0$

$\alpha\beta\gamma=-\frac{d}{a}$

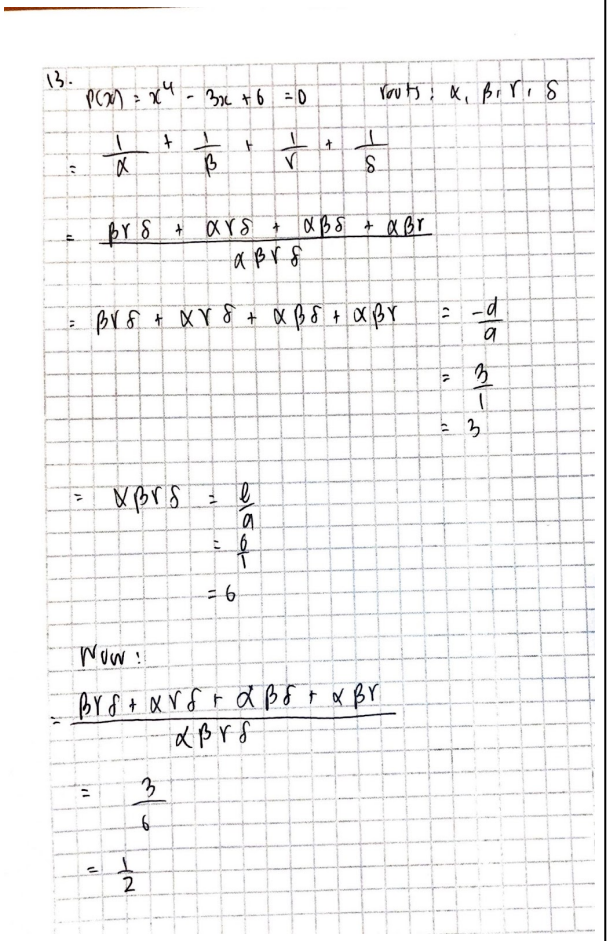
$=\frac{1}{2}$

$=\frac{1}{\left(\frac{1}{2}\right)^3}$

$=1\div\frac{1}{8}$

$=8$

Ans: C



13. $P(x)=x^4-3x+6=0$ roots: $\alpha,\beta,\gamma,\delta$

$=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$

$=\frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta}$

$=\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma \quad =\frac{-d}{a}$

$=\frac{3}{1}$

$=3$

$=\alpha\beta\gamma\delta=\frac{e}{a}$

$=\frac{6}{1}$

$=6$

Now:

$=\frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta}$

$=\frac{3}{6}$

$=\frac{1}{2}$

14. Functions, EXT1 F2 2013 HSC 11a

The polynomial equation $2x^3-3x^2-11x+7=0$ has roots α, β and γ.

Find $\alpha\beta\gamma$. *(1 mark)*

20. Functions, EXT1 F2 SM-Bank 2

The polynomial $P(x)=x^3-2x^2+kx+24$ has roots α,β,γ.

i. Find the value of $\alpha+\beta+\gamma$. *(1 mark)*

ii. Find the value of $\alpha\beta\gamma$. *(1 mark)*

iii. It is known that two of the roots are equal in magnitude but opposite in sign. Find the third root and hence find the value of k. *(2 marks)*

25. Functions, EXT1' F2 2009 HSC 3c

Let $P(x)=x^3+ax^2+bx+5$, where a and b are real numbers.

Find the values of a and b given that $(x-1)^2$ is a factor of $P(x)$. *(3 marks)*

14. $P(x) = x^4 + ax^3 + bx^2 + cx + 1$

roots: $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$

$\gamma \quad \delta$

Identity: $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$

& $\alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = -\frac{a}{1}$

~~$\alpha\beta + \alpha\frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha}\beta + \frac{1}{\alpha}\frac{1}{\beta} + \alpha\beta + \frac{1}{\beta}\beta = \frac{b}{a}$~~

$\alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = -a$

$\frac{\alpha^2}{\alpha} + \frac{1}{\alpha} + \frac{\beta^2}{\beta} + \frac{1}{\beta} = -\frac{b}{a}$

~~$\alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = -\frac{b}{a}$~~

~~Identity: $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$~~ need to use identity with 6

$\alpha\beta + \frac{\alpha}{\beta} + \frac{\alpha}{\alpha} + \frac{\beta}{\alpha} + \frac{\beta}{\beta} + \frac{1}{\alpha\beta} = \frac{c}{a}$

$\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} + 2 = \frac{c}{a}$

Identity: $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$

$\frac{\alpha\beta}{\alpha} + \frac{\alpha\beta}{\beta} + \frac{\alpha}{\alpha\beta} + \frac{\beta}{\alpha\beta} = -\frac{c}{1}$

$\alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = -c$

$\therefore -a = -c$

$a = c$ (as required)

20.

i) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= \frac{2}{1}$

$= 2$

ii) $\alpha\beta\gamma = -\frac{d}{a}$

$= -\frac{24}{1}$

$= -24$

iii) Let $\beta = -\alpha$

$\therefore \alpha + (-\alpha) + \gamma = 2$

$\gamma = 2$

~~$\alpha(-\alpha)\gamma = -24$~~

~~$-\alpha^2(2) = -24$~~

since γ is a root:

$P(\gamma) = (\gamma)^3 - 2(\gamma)^2 + k(\gamma) + 24 = 0$

$= (2)^3 - 2(2)^2 + k(2) + 24 = 0$

$= 8 - 8 + 2k + 24 = 0$

$\boxed{k = -12}$

25. $P(x) = x^3 + ax^2 + bx + 5$

given that $(x-1)^2$ is a factor

$P(1) = (1)^3 + a(1)^2 + b(1) + 5$

$= 0$

$1 + a + b + 5 = 0$

$a + b = -6$ ①

~~$\alpha + \beta + \gamma = -\frac{b}{a}$~~

~~$= -a$~~

$P'(x) = 3x^2 + 2ax + b$

$P'(1) = 3(1)^2 + 2a(1) + b$

$= 0$

$3 + 2a + b = 0$

$2a + b = -3$ ②

① - ②

$a + b = -6$

$2a + b = -3$

$-a = -3$

$\boxed{a = 3}$

Sub $a = 3$ into ①

$3 + b = -6$

$\boxed{b = -9}$

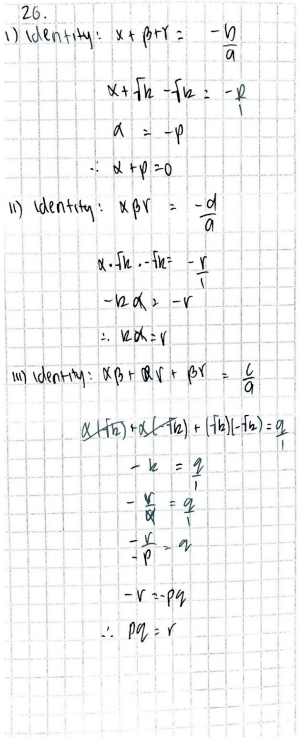
26. Functions, EXT1 F2 SM-Bank 5

The polynomial $P(x) = x^3 + px^2 + qx + r$ has roots $\sqrt{k}, -\sqrt{k}$ and α.

i. Explain why $\alpha + p = 0$. *(1 mark)*

ii. Show that $k\alpha = r$. *(1 mark)*

iii. Show that $pq = r$. *(2 marks)*



26.

i) Identity: $\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha + \sqrt{k} - \sqrt{k} = -\frac{p}{1}$

$\alpha = -p$

$\therefore \alpha + p = 0$

ii) Identity: $\alpha\beta\gamma = -\frac{d}{a}$

$\alpha \cdot \sqrt{k} \cdot -\sqrt{k} = -\frac{r}{1}$

$-k\alpha = -r$

$\therefore k\alpha = r$

iii) Identity: $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$\alpha(\sqrt{k}) + \alpha(-\sqrt{k}) + (\sqrt{k})(-\sqrt{k}) = \frac{q}{1}$

$-k = \frac{q}{1}$

$-\frac{r}{\alpha} = \frac{q}{1}$

$-\frac{r}{-p} = q$

$-r = -pq$

$\therefore pq = r$

Conclusion

This lesson was very insightful because it had allowed me to explore how calculus can help with understanding polynomials, and how these past hsc exams indeed help in aiding how one approaches a question. This lesson had taught me a lot about exam techniques, approach and how to link other concepts from other topics of maths to answer a question.

Syllabus Outcomes: achieved
Learning Goal: Achieved

Lesson 6: Multiplicity of Roots

Syllabus Dot Points:

- determine the multiplicity of a root of a polynomial equation
    - prove that if a polynomial equation of the form $P(x) = 0$ has a root of multiplicity $r > 1$, then $P'(x) = 0$ has a root of multiplicity $r - 1$
- graph a variety of polynomials and investigate the link between the root of a polynomial equation and the zero on the graph of the related polynomial function
    - examine the sign change of the function and shape of the graph either side of roots of varying multiplicity

Learning Goals:

- ☑ ~~Understand what the multiplicity of roots tells us~~
- ☑ ~~Determine the multiplicity of a root of a polynomial~~
- ☑ ~~Graph functions with different multiplicities~~

Questions

1 Consider the polynomial $P(x) = x^3 - 4x^2 - 3x + 18$.

a i Show that $P(3)$ and $P'(3)$ are both zero.

ii What can be deduced from the results in part **i**?

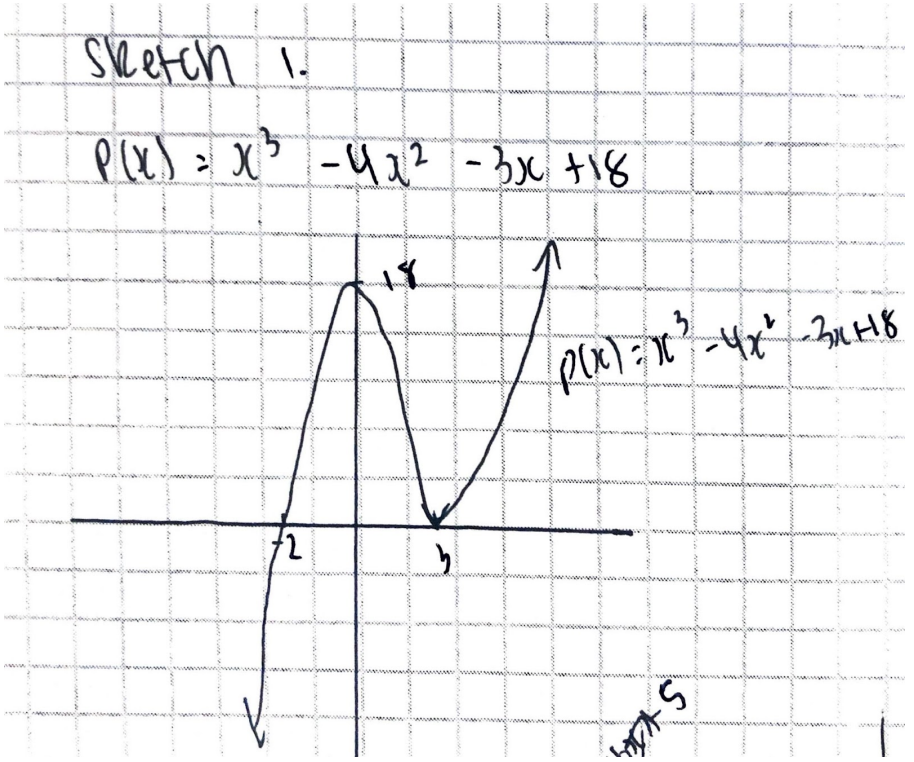
1. $P(x) = x^3 - 4x^2 - 3x + 18$

a) $P(3) = (3)^3 - 4(3)^2 - 3(3) + 18$
$= 27 - 36 - 9 + 18$
$= 0$

$P'(x) = 3x^2 - 8x - 3$
$P'(3) = 3(3)^2 - 8(3) - 3$
$= 27 - 24 - 3$
$= 0$

tells us that @ $x = 3$, there is a turning point, where $x = 3$ is also a double root

Sketch of this polynomial



2 Consider the polynomial $P(x) = x^4 + 8x^3 + 18x^2 + 16x + 5$.

a i Show that $P(-1)$, $P'(-1)$ and $P''(-1)$ are all zero.

ii What can be deduced from the results in part **i**?

Sketch of this polynomial

2. $P(x) = x^4 + 8x^3 + 15x^2 + 16x + 5$

a)

$P(-1) = (-1)^4 + 8(-1)^3 + 15(-1)^2 + 16(-1) + 5$

$= 0$

$P'(x) = 4x^3 + 24x^2 + 30x + 16$

$P'(-1) = 4(-1)^3 + 24(-1)^2 + 36(-1) + 16$

$= 0$

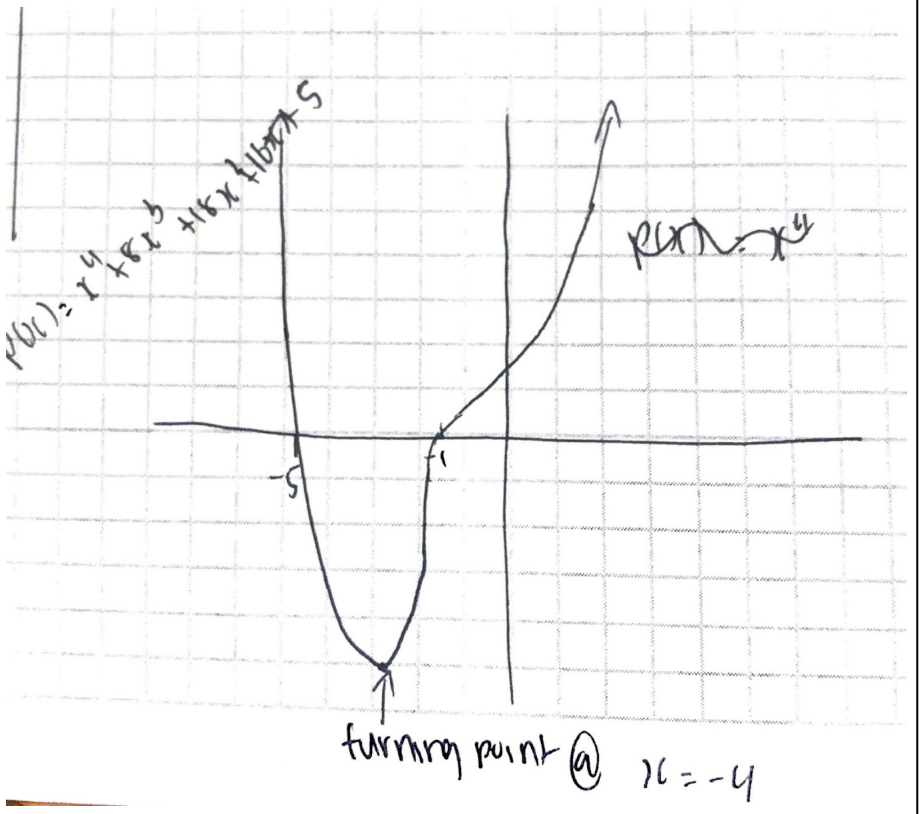
$P''(x) = 12x^2 + 48x + 36$

$P''(-1) = 12(-1)^2 + 48(-1) + 36$

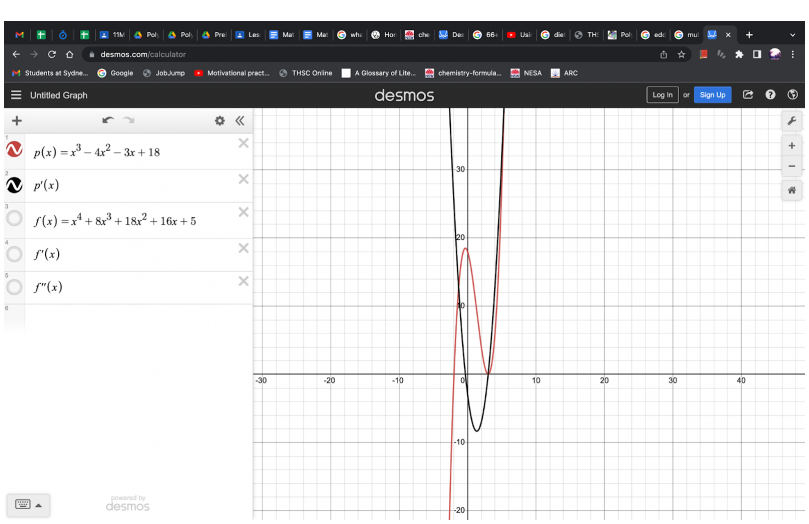
$= 12 - 48 + 36$

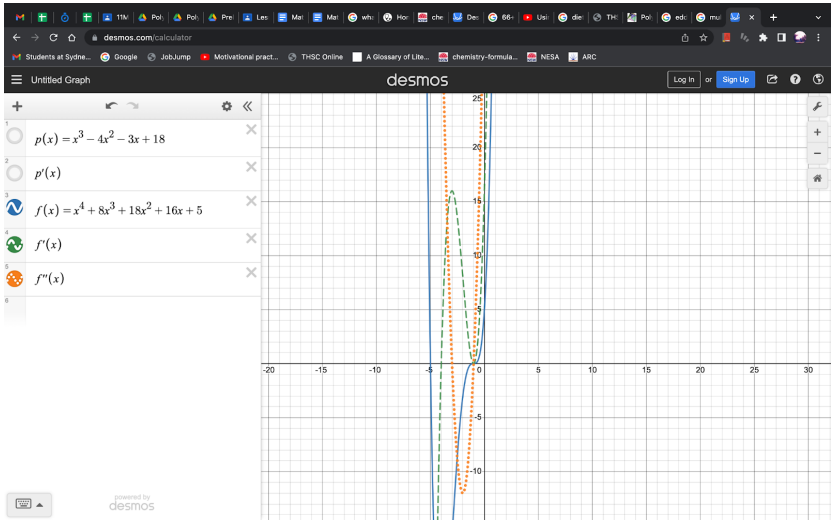
$= 0$

tells us that @ $x = -1$, it is a triple root of $P(x)$



Further Insight





These images show how the derivative and how the multiplicity of roots are connected and applied, which then deepens my understanding of polynomials and derivatives and how it can help with understanding the multiplicity of roots.

Conclusion

This lesson was a very insightful lesson that helped me connect my knowledge about calculus with polynomials in a way that was very helpful to visualise through the help of desmos. It can be seen that the multiplicity of roots depends on the derivative of the function, and essentially how many times that point passes through the polynomial along with the derivatives

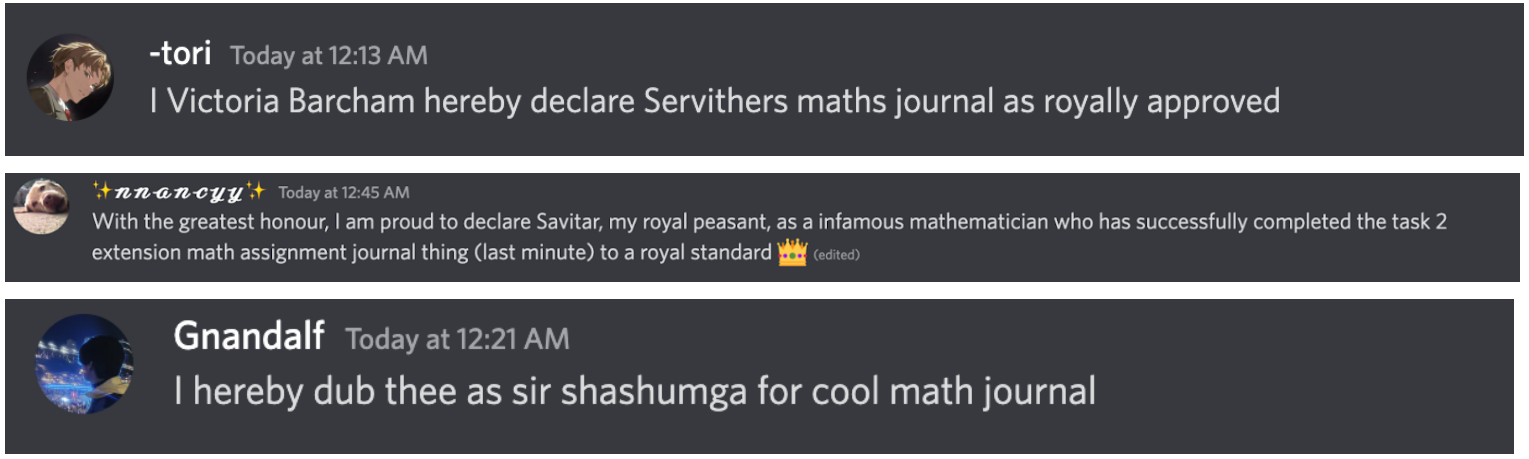
Syllabus Outcome: Achieved
Learning Goal: Achieved

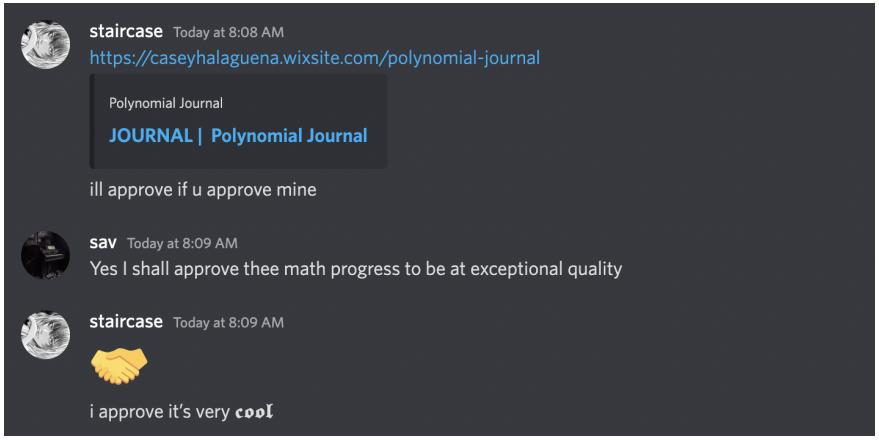
Conclusion of the Journal

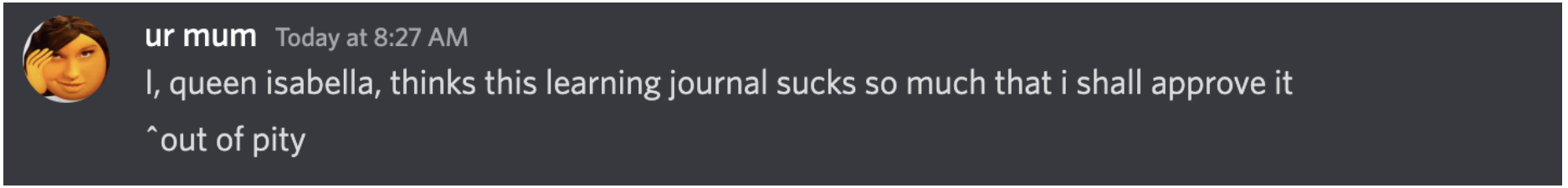
Therefore this journey had really helped me explore the world of maths in a different lens as it allowed me to connect ideas from different topics to broaden how functions and polynomials behave.

Credibility

To assess the credibility of this journal, I have added important personnel that approve of this journal, and hence shows the accuracy of the information of this journal:







Thank you, and have a great day :))